Extensions of generalized product caps

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Abstract

We give some variants of a new construction for caps. As an application of these constructions we obtain a 1216–cap in $PG(9,3)$ a 6464–cap in $PG(11,3)$ and several caps in ternary affine spaces of larger dimension, which lead to better asymptotics than the caps constructed by Calderbank and Fishburn [1]. These asymptotic improvements become visible in dimensions as low as 62, whereas the bound from [1] is based on caps in dimension $13,500$.

1 Introduction

Let $PG(n,q)$ be the projective space of dimension n over the finite field $\mathbb{F}_q$. A k–**cap** K in $PG(n,q)$ is a set of k points, no three of which are collinear [10]. The maximum value of k for which there exists a k–cap in $PG(n,q)$ is denoted by $m_2(n,q)$. Denote by $m_2^{aff}(n,q)$ the corresponding value in $AG(n,q)$. As $m_2(n,2) = m_2^{aff}(n,2) = 2^n$ we can and will assume $q > 2$ in the sequel. The numbers $m_2(n,q)$, $m_2^{aff}(n,q)$ are only known, for arbitrary q, when $n \in \{2,3\}$, namely, $m_2(2,q) = m_2^{aff}(2,q) = q+1$ if q is odd, $m_2(2,q) = m_2^{aff}(2,q) = q+2$ if q is even, and $m_2(3,q) = q^2+1$, $m_2^{aff}(3,q) = q^2$. Aside of these general results the precise values are known only in the following cases: $m_2(4,3) = m_2^{aff}(4,3) = 20$ [13], $m_2(5,3) = 56$ [7], $m_2^{aff}(5,3) = 45$ [5], and $m_2(4,4) = 41$ [3]. Finding the exact value for $m_2(n,q)$ or $m_2^{aff}(n,q)$, $n \geq 4$ seems to be a very hard problem [8, 9]. As an application of our new construction we obtain improved lower bounds on some values $m_2(n,3)$. The first examples of improvements are a 1216–cap in $PG(9,3)$ and a 6464–cap in $PG(11,3)$.

A natural asymptotic problem is the determination of

$$\mu(q) = \limsup_{n\to\infty} \frac{\log_q(m_2(n,q))}{n} = \limsup_{n\to\infty} \frac{\log_q(m_2^{aff}(n,q))}{n}.$$

It is well-known (and also will be explained later) that for every cap A in $AG(n,q)$ we have the inequality $\mu(q) \geq \log_q(|A|)/n$. As a cap cannot be larger than its ambient space, clearly $\mu(q) \leq 1$. It is an interesting open problem to decide if $\mu(q) < 1$. The affine part of an ovoid in $PG(3,q)$ shows $\mu(q) \geq \frac{2}{3}$. The affine points of a family of caps in $PG(6,q)$ from [2] yield the slightly better bound $\mu(q) \geq \frac{\log_q(q^4+q^2-1)}{6}$. No better lower bound seems to be known for general q, except for the ternary and quaternary cases. It follows from [1] that $\mu(3) \geq 0.7218\ldots$. The 120 affine points of the 126-cap in $PG(5,4)$ found by Glynn [4, 6] show that $\mu(4) \geq 0.3 + \frac{\log_4(15)}{5} = 0.6906\ldots$ The construction given in this article can be seen as a generalization of one of the constructions of Calderbank and Fishburn [1]. Although the construction works for general q all our applications are in the ternary case. Our constructions of caps in ternary affine spaces lead to a better bound for $\mu(3)$. The best bound proved in this article is $\mu(3) \geq 0.724851\ldots$.

This leaves us with two research problems. Firstly to improve the bound on $\mu(3)$ by finding better capsets (for a definition see Definition 9), secondly to find good caps to which we can apply the construction for $q > 3$.

2 The product construction

A cap $A \subset AG(n,q)$ is a subset $A \subset \mathbb{F}_q^n$ such that the points $(1:a)$, $a \in A$ form a cap in $PG(n,q)$. Let $B \subset \mathbb{F}_q^{m+1}$ be a set of representatives of a cap in $PG(m,q)$. For every $0 \neq a \in \mathbb{F}_q^n$ denote by $\langle a\rangle$ the 1-dimensional subspace $\mathbb{F}_q a$. This is a point in $PG(n-1,q)$. For $0 \notin A \subset AG(n,q)$ denote $\langle A\rangle = \{\langle a\rangle | a \in A\}$.

Theorem 1 (the product construction)**.** *Let $A \subset AG(n,q)$ be a cap and $B \subset \mathbb{F}_q^{m+1}$ be a set of representatives of a cap $\langle B\rangle \subset PG(m,q)$. Then $(A:B) := \{(a:b)|a \in A, b \in B\} \subset PG(n+m,q)$ is an $(|A|\cdot|B|)$-cap. If $\langle B\rangle \subset AG(m,q)$, then $(A:B) \subset AG(n+m,q)$.*

Theorem 1 is due to Mukhopadhyay [12]. The special case when $n = 1$ and A consists of two points in $AG(1, q) = \mathbb{F}_q$ yields a $2|B|$-cap in $AG(m + 1, q)$. This is the well-known **doubling construction**. The following is a generalization of the product construction:

Theorem 2 (generalized product construction)**.** *Let $A_1, \ldots, A_c \subset AG(n, q)$ be caps and $B \subset \mathbb{F}_q^{m+1}$ a set of representatives of a cap $\langle B\rangle \subset PG(m, q)$, partitioned as $B = B_1 \cup \cdots \cup B_c$. Then $\bigcup_{i=1}^{c}(A_i : B_i)$ is a cap in $PG(n+m, q)$.*

Proof. Each $(A_i : B_i)$ is a cap by Theorem 1. The second coordinate shows that the union is a cap. ∎

If in Theorem 2 we choose $A_1 = A_2 = \cdots = A_c$ Theorem 1 is obtained. We study the generalized product construction in the hope to obtain products which are not complete.

Let a generalized product cap be given. We ask when a point $(u : v)$ will be an extension point. The case $v = 0$ is easily decided:

Theorem 3. *A point $(u : 0)$ extends the generalized product cap (Theorem 2) if and only if $(0 : u)$ extends all affine caps $(1 : A_i)$.*

Proof. $(u : 0)$ is not an extension point if and only if we have a relation

$$(u, 0) = \lambda(a, b) + \lambda'(a', b').$$

We must have $b = b' \in B_i$ for some i and $\lambda+\lambda' = 0$, equivalently $u = \lambda(a-a')$, where $a, a' \in A_i$ for some i. This is precisely the condition that $(0 : u)$ is not an extension point of $(1 : A_i)$. ∎

We see that in the situation of Theorem 3 the generalization of the product construction presents no advantage, as we get caps of same size with the ordinary product construction (Theorem 1) by choosing $A_1 = \cdots = A_c$. Based on Theorem 1, Theorem 3 leads to a generalization of the product construction, which we proved in [2]. An application to ovoids yields Segre's recursive construction [14].

Consider now points $(u : v)$, where $v \neq 0$. Assume also $u \neq 0$. Such a point is not an extension point if and only if there is a relation

$$(u, v) = \lambda(a, b) + \lambda'(a', b')$$

The following strategy will make sure this cannot happen:

- Choose $u \neq 0$ and the A_i such that $\langle u \rangle \notin \langle A_i \rangle$ for all i and such that for all $a \in A_i$, $a' \in A_j$, $i \neq j, \langle a \rangle \neq \langle a' \rangle$, the points $\langle u \rangle$, $\langle a \rangle$, $\langle a' \rangle$ are not collinear. In the above relation this forces $\{a, a'\} \subseteq A_i$ for some i.

- Choose $v \neq 0$ such that $\langle v \rangle \notin \langle B \rangle$ and $\langle B_i \rangle \cup \{\langle v \rangle\}$ is a cap for all i.

If these conditions are satisfied, then $(u : v)$ is an extension point. The first condition will be easier to satisfy for a small number of components (small c), the second condition is easier to satisfy when c is large. Let now $A_0 \subset AG(n,q) \setminus \{0\}$ be a cap and $B_0 \subset \mathbb{F}_q^{m+1}$ a set of representatives of a cap such that $(u : v)$ satisfies the conditions above for all $u \in A_0$, $v \in B_0$. Then $(A_0 : B_0)$ is a cap by Theorem 1. We want the union of the generalized product cap and $(A_0 : B_0)$ to be a cap. It remains to make sure that two different points of $(A_0 : B_0)$ can never be collinear with a point from the generalized product cap. A sufficient condition is that for every $i \neq 0$ no two different points of $\langle A_0 \rangle$ are collinear with a point from $\langle A_i \rangle$. This motivates the following definition:

Definition 4. *Let $A_i \subset AG(n,q)$, $i = 0, \ldots, c$, be caps, where $0 \notin A_i$. We say that $(A_0, \{A_i\}_{i=1}^c)$ satisfy property (E_L) if the following hold:*

(1) *$\langle A_0 \rangle \cap \langle A_i \rangle = \emptyset$ for all $i > 0$,*

(2) *If $a_i \in A_i$, then $\langle a_i \rangle$ is not collinear with two different points of $\langle A_0 \rangle$.*

(3) *If $u \in A_0$, $a \in A_i$, $a' \in A_j$, $i \neq j$; $i, j > 0$, $\langle a \rangle \neq \langle a' \rangle$, then $\langle u \rangle$, $\langle a \rangle$, $\langle a' \rangle$ are not collinear.*

Let $B \subset PG(m,q)$ be a system of representatives of a cap $\langle B \rangle \subset PG(m,q)$, partitioned in the form $B = B_1 \cup \cdots \cup B_c$, and $B_0 \subset PG(m,q)$ a system of representatives of a cap $\langle B_0 \rangle$, which is disjoint from $\langle B \rangle$ and such that $\langle B_i \rangle \cup \{\langle v \rangle\}$ is a cap for all $i > 0$ and all $v \in B_0$. We say that $(B_0, \{B_i\}_{i=1}^c)$ satisfy property (E_R).

Observe that it can happen that two different elements $u \neq u'$ of A_0 are scalar multiples of each other and therefore give rise to the same point $\langle u \rangle = \langle u' \rangle \in PG(n-1,q)$. Note also that $\langle A_0 \rangle$ need not be a cap in $PG(n-1,q)$.

We have proved the following above:

Theorem 5. *Let $0 \notin A_i \subset AG(n,q)$, $i = 0, 1, \ldots, c$ be caps such that (E_L) is satisfied. Let $B_0, B \subset \mathbb{F}_q^{m+1}$ be systems of representatives of caps, $B = B_1 \cup \cdots \cup B_c$, satisfying (E_R). Then $K = \bigcup_{i=0}^{c}(A_i : B_i) \subset PG(n+m,q)$ is a cap. If both $\langle B_0 \rangle$ and $\langle B \rangle$ are contained in $AG(m,q)$ (equivalently: avoiding a hyperplane $H \subset PG(m,q)$) or the A_i are avoiding a hyperplane of $AG(n,q)$ (different from the one at infinity), then $K \subset AG(n+m,q)$.*

It is a strength of Theorem 5 that the components can be constructed separately. The cap constructed in Theorem 5 has $\sum_{i=0}^{c} |A_i||B_i|$ points. If all A_i, $i > 0$, have equal size $|A|$ this simplifies to $|A_0||B_0| + |A||B|$.

3 The case of the doubled Hill cap

Particularly fruitful applications of Theorem 5 are obtained when $q = 3$, $n = 6$ and A_1, A_2 are two versions of the doubled Hill cap.

Definition 6. *Consider the following subsets of $\mathbb{F}_3^6$: D consists of the weight 3 vectors whose supports form the blocks of a fixed 2-$(6,3,2)$ design, D' consists of the remaining vectors of weight 3. Let R be the vectors of weight 6 with an even number of entries 2 and R' the remaining vectors of weight 6. Also, A_0 consists of the vectors of weight 1. Finally*

$$H = D \cup R \text{ and } H' = D' \cup R.$$

Then both H and H' are versions of the doubled Hill cap [4, 1] (a 112-cap in $AG(6,3)$). We use $A_1 = H$, $A_2 = H'$. Observe $|A_1 \cap A_2| = 32$.

Lemma 7. $H + H' = \mathbb{F}_3^6 \setminus A_0$

Proof. It is in fact clear that elements of weight 1 are not in $D + R$ or $D' + R$ or $D + D'$. A routine check shows that all other elements have one of these forms. ■

Observe that A_0 itself is a doubled cap and hence a 12-cap in $AG(6,3)$. We can use A_0 in Theorem 5. It remains to find caps $\langle B \rangle$ in $PG(m,q)$ or in $AG(m,q)$, to partition them into two suitable parts and to find sets B_0.

The smallest case is $m = 1$. Both B_1 and B_2 consist of one point, B_0 has one element in the affine case, two elements in the projective case. Theorem 5 yields a 236-cap in $AG(7,3)$ (see [1]) and a 248-cap in $PG(7,3)$ (see [4]).

Consider case $m = 3$. We wish to partition the ovoid into two parts. Describe the field $\mathbb{F}_9$ by the polynomial $X^2 - X - 1$, in other words $\mathbb{F}_9 = \mathbb{F}_3(\epsilon)$, where $\epsilon^2 = \epsilon + 1$. Represent the affine points of the ovoid as $(x : N(x) : 1)$, where $x \in \mathbb{F}_9$ and $N(x) = x^4 \in \mathbb{F}_3$. Let $Q = \{\pm 1, \pm \epsilon^2\}$ (the squares) and $N = \{\pm \epsilon, \pm(\epsilon - 1)\}$ (the nonsquares in $\mathbb{F}_9$). The affine points of the ovoid therefore have the forms $(0 : 0 : 1)$, $(Q : 1 : 1)$, $(N : 2 : 1)$, the point at infinity is $(0 : 1 : 0)$ (here the first coordinate represents two coordinates). Choose

$$B_1 = \{(0, 0, 1)\} \cup \{(Q, 1, 1)\} \text{ and } B_2 = \{(0, 1, 0)\} \cup \{(N, 2, 1)\}$$

It is easy to see that the points which form extensions both of $\langle B_1 \rangle$ and of $\langle B_2 \rangle$ are the eight points of the form

$$(Q : 0 : 1) \text{ and } (N : 1 : 0)$$

These extension points form an 8-cap. Theorem 5 yields a cap of size $112 \cdot 10 + 12 \cdot 8 = 1216$ in $PG(9, 3)$.

Here is an application when $m = 5$. We choose B to be a set of representatives of the Hill cap, partitioned such that $\langle B_1 \rangle = \langle R \rangle$ and $\langle B_2 \rangle = \langle D \rangle$. It is clear that the 16 point from $\langle R' \rangle$ form an extension cap of $\langle B_1 \rangle$ and of $\langle B_2 \rangle$. This yields a cap of size $112 \cdot 56 + 12 \cdot 16 = 6464$ in $PG(11, 3)$.

4 Recursive constructions

Next we give a recursive construction for caps which satisfy (E_L).

Definition 8. *Let $A \subset \mathbb{F}_q^n = AG(n, q)$ and $A^l := (A, A, \ldots, A) \subset AG(nl, q)$. For $s = (s_1, \ldots, s_l) \in \{0, \ldots, c\}^l$ and $A_i \subset AG(n, q)$ define*

$$s(A_0, \ldots, A_c) := (A_{s_1}, \ldots, A_{s_l}) \subset AG(ln, q).$$

For $S \subset \{0, \ldots, c\}^l$ define

$$S(A_0, \ldots, A_c) := \bigcup_{s \in S} s(A_0, \ldots, A_c)$$

Definition 9. *We say $S \subset \{0, \ldots, c\}^l$ is a* **capset** *if the following are satisfied:*

(1) *for every pair* $s \neq s' \in S$ *there is a coordinate* i *where* $s_i = 0 \neq s'_i$ *and a coordinate* j *where* $s_j \neq 0 = s'_j$.

(2) *for every triple of distinct* $s, s', s'' \in S$ *there is a coordinate* i *such that* $\{s_i, s'_i, s''_i\}$ *is either* $\{0, u, v\}$ *or* $\{0, 0, u\}$, *with* $u \neq v \in \{1, \ldots, c\}$.

Let S *be a capset. We say* S *is an* **admissible set** *if in addition* $|S| \geq 2$, $l \geq 2$ *and for every pair* $s \neq s' \in S$ *at least one of the two following properties is satisfied*

(3) *there is a coordinate* i *where* $\{s_i, s'_i\} = \{0, u\}$ *and a coordinate* j *where* $\{s_j, s'_j\} = \{0, v\}$, *with* $u \neq v \in \{1, \ldots, c\}$, *or*

(4) *there is a coordinate* i *where* $s_i = s'_i = 0$.

The motivation for Definition 9 is the following lemma:

Lemma 10. *Let* $(A_0, \{A_i\}_{i=1}^c)$ *satisfy property* (E_L). *If* $S \subset \{0, \ldots, c\}^l$ *is a capset then* $S(A_0, \ldots, A_c)$ *is a cap in* $AG(ln, q)$.

If S *is an admissible set, then* $(S(A_0, \ldots, A_c), \{A_i^l\}_{i=1}^c)$ *satisfies property* (E_L).

If ν_i is the frequency of the entry i in s then $s(A_0, \ldots, A_c)$ contains $\prod_i |A_i|^{\nu_i}$ points. In our examples we will have the situation that all $|A_i| = N$ for $i > 0$ and all $s \in S$ have equal weight w. In this case the number of points in $S(A_0, \ldots, A_c)$ is $|S| N^w |A_0|^{l-w}$.

Proof. Assume S is a capset. Theorem 1 shows that $s(A_0, \ldots, A_c)$ is a cap for all s. Let $s, s' \in S$, $s \neq s'$. We want to show that the union $s(A_0, \ldots, A_c) \cup s'(A_0, \ldots, A_c)$ of two blocks is a cap. Assume without restriction that two points from $s(A_0, \ldots, A_c)$ are collinear with a point from $s'(A_0, \ldots, A_c)$. By Property (1) there is a coordinate section where each of the points from $s(A_0, \ldots, A_c)$ projects to an element from A_0 and the third point projects to an element from A_i for some $i \neq 0$. This means there exist nonzero coefficients λ_1, λ_2, λ_3, $\sum_{i=1}^3 \lambda_i = 0$, and elements $a_0, a'_0 \in A_0$, $a_i \in A_i$ such that $\lambda_1 a_0 + \lambda_2 a'_0 + \lambda_3 a_i = 0$. If $\langle a_0 \rangle = \langle a'_0 \rangle$ then $(E_L(1))$ yields a contradiction, $(E_L(2))$ yields a contradiction if $\langle a_0 \rangle \neq \langle a'_0 \rangle$.

Likewise, Property (2) shows that the union of three blocks is a cap, in the first alternative by using $(E_L(1))$ or $(E_L(3))$, making use of $(E_L(1))$ or $(E_L(2))$ in the second alternative.

Assume now S is an admissible set. We show that condition (E_L) is satisfied.

$(E_L(1))$ follows from (1), as for every $s \in S$ there is a coordinate i such that $s_i = 0$, by using $(E_L(1))$ of $(A_0, \{A_i\}_{i=1}^c)$.

$(E_L(2))$: Assume $\langle a \rangle$, $a \in A_i^l$, $i \neq 0$ is collinear with $\langle x \rangle$, $x \in s(A_0, \ldots, A_c)$ and $\langle y \rangle$, $y \in s'(A_0, \ldots, A_c)$. If $s = s'$, a coordinate where $s_i = 0$ yields a contradiction because of $(E_L(1))$ or $(E_L(2))$. If $s \neq s'$, we use admissibility. In case of (3) use $(E_L(1))$ or $(E_L(3))$, in case of (4) use $(E_L(1))$ or $(E_L(2))$ to obtain a contradiction.

$(E_L(3))$: Properties $(E_L(1))$ and $(E_L(3))$ of $(A_0, \{A_i\}_{i=1}^c)$ show that points $\langle a \rangle, \langle a' \rangle, \langle x \rangle$ cannot be collinear when $a \in A_i^l$, $a' \in A_j^l$ for $i \neq j$; $i, j \neq 0$ and $x \in s(A_0, \ldots, A_c)$, $s \in S$ as there is a coordinate i where $s_i = 0$. ■

Lemma 10 can be generalized in an obvious way, using different caps $(A_0^{(j)}, \{A_i^{(j)}\}_{i=1}^c) \subset AG(n_j, q)$ for each coordinate section j, $1 \leq j \leq l$. We will not make use of this generalization here.

The following lemma is obvious:

Lemma 11. *Let S be a capset, let $(A_0, \{A_i\}_{i=1}^c)$ satisfy property (E_L) and $\Delta = A_i \cap A_j$, $i \neq j$. Then $S(A_0, \ldots, A_c) \cup \Delta^l$ is a cap in $AG(ln, q)$.*

Now it is high time to give some examples of capsets and admissible sets.

Definition 12. *Denote by $I_c(l, t)$ an admissible set in $\{0, \ldots, c\}^l$ consisting of $\binom{l}{t}$ vectors of weight $l - t$ and by $\tilde{I}_c(l, t)$ a capset of this type.*

Lemma 13. *There exists an $I_c(l, c-1)$ for all $l > c$.*

Proof. Define this set of vectors as the $\binom{l}{c-1}$ vectors of weight $l - c + 1$ with entries $i + 1$ between the i-th and $i + 1$-th zero (if any) and with entries 1 before the first zero, entry c after the last zero, if any.

As all vectors have different support, condition (1) of Definition 9 is automatically fulfilled. Now consider condition (2). Consider three different vectors s, s', s'' of $I_c(l, c-1)$. We can assume that there is no coordinate i with $\{s_i, s_i', s_i''\} = \{0, 0, u\}$. As the vectors have different support there is a first coordinate i where exactly one of the s_i, s_i', s_i'' is zero. We may assume that $s_i = 0$. Let j be the first coordinate where $s_j \neq 0$ and s_j' or s_j'' is zero. We may assume that $s_j' = 0$. So there are more zeroes in s up to coordinate j than in s''. It follows $s_j > s_j'' > 0$, hence condition (2) is satisfied.

Let s, s' be two different vectors from our set. Assume that condition (4) is not satisfied. In particular there is no coordinate i with $s_i = s'_i = 0$. Let s be the vector with the smallest coordinate where a zero appears, let this coordinate be i. Let j be the first coordinate where a zero appears in s'. We have $s'_i = 1$ and $s_j > 1$, so condition (3) is satisfied. ■

The first series of Calderbank and Fishburn [1] is obtained applying Lemma 11 with A_0, A_1, A_2 from the doubled Hill cap as introduced in Section 3, and $S = I_2(l, 1)$.

The following vectors $s = (s_1, \ldots, s_{10})$ and their cyclic shifts form an $\tilde{I}_2(10, 5)$. Observe that the orbit of the last vector has only length 2.

$$\begin{array}{ll}
(0,0,0,0,0,1,1,1,1,1) & (0,0,0,0,1,0,1,1,1,2) \\
(0,0,0,0,1,2,0,1,1,2) & (0,0,0,0,1,2,2,0,1,2) \\
(0,0,0,0,1,2,2,2,0,2) & (0,0,0,1,0,0,1,1,1,2) \\
(0,0,0,1,0,2,0,1,1,2) & (0,0,0,2,0,1,1,0,2,1) \\
(0,0,0,1,0,2,2,2,0,1) & (0,0,0,2,1,0,0,1,1,2) \\
(0,0,0,2,1,0,2,0,1,2) & (0,0,0,1,2,0,1,1,0,2) \\
(0,0,0,2,1,1,0,0,2,2) & (0,0,0,1,2,2,0,2,0,2) \\
(0,0,0,1,2,2,1,0,0,2) & (0,0,2,0,0,2,0,1,1,1) \\
(0,0,1,0,0,2,1,0,1,2) & (0,0,1,0,0,2,1,2,0,1) \\
(0,0,2,0,2,0,0,2,1,2) & (0,0,2,0,1,0,1,0,1,2) \\
(0,0,1,0,2,0,1,1,0,2) & (0,0,2,0,2,2,0,0,1,2) \\
(0,0,2,0,2,1,0,2,0,1) & (0,0,2,1,0,0,2,2,0,2) \\
(0,0,1,1,0,2,0,2,0,1) & (0,2,0,2,0,2,0,2,0,2)
\end{array}$$

Also, $I_2(9, 2)$, $I_2(10, 3)$, $I_2(9, 4)$, $I_2(9, 5)$, $I_2(10, 6)$ and $\tilde{I}_2(11, 2)$ were found by computer and are available on the author's homepage [15].

5 Asymptotic results

It follows from Theorem 1 that $\mu(q) \geq \log_q(|A|)/n$ for every cap A in $AG(n, q)$. In the ternary case the lower bound from Calderbank and Fishburn [1] is $\mu(3) \geq 0.7218\ldots$. It is based on a cap in $AG(13500, 3)$ (the doubled Hill cap yields $\mu(3) \geq 0.7158\ldots$)

Our first asymptotic improvement happens in $AG(62, 3)$. Apply Lemma10 with $n = 6$, $c = 2$, where A_0, A_1, A_2 are derived from the doubled Hill cap in $AG(6, 3)$ as in Section 3 (recall $|A_1| = |A_2| = 112$, $|A_0| = 12$). As admissible set choose $I_2(l,1)$ (see Lemma 13). The result is a cap in $AG(6l,3)$. Apply Theorem 5 with $n=6l$, $m=1$, where the B_i are from the projective case as in Section 3 ($|B_1|=|B_2|=1$, $|B_0|=2$). The result is a cap in $PG(6l+1,3)$. The final result is obtained by applying the doubling construction. The asymptotic expression has its maximum at $l=10$. We have a cap in $AG(62,3)$. The number of its points is $2*(2*112^{10}+2*10*112^9*12)$, yielding $\mu(3)\geq 0.723779\ldots$.

The use of different values of m as in Section 3 produces further examples of good caps but no asymptotic improvement.

Let us apply Lemma 10 recursively. Start from the admissible set $S\subset\{0,1,\ldots,c\}^l$. For simplicity assume $|A_i|=N$ for all $i\neq 0$, $|A_0|=M$ and that all elements of S have the same weight $l-s_0$. It follows from Lemma 10 that the family of caps $(S(A_0,\ldots,A_c),\{A_i^l\}_{i=1}^c)$ in $AG(ln,q)$ satisfies property (E_L). Apply Lemma 10 again, with a capset $T\subset\{0,1,\ldots,c\}^k$, all of whose elements have weight $k-t_0$. The result is a cap in $AG(kln,q)$, which we denote for simplicity as $T(S(A))$, where $A=(A_0,\{A_i\}_{i=1}^c)$. We have

$$|T(S(A))|=|T|\cdot|S|^{t_0}N^{lk-t_0s_0}M^{s_0t_0}.$$

In our favorite ternary case ($n=6$, $c=2$, $N=112$, $M=12$) we use $S=I_2(8,1)$ and T the $\tilde{I}_2(10,5)$ constructed in Section 4. Finally we can apply Lemma 11 with $\Delta=A_1^l\cap A_2^l$, $|\Delta^k|=32^{kl}=32^{80}$. We have constructed a cap in $AG(480,3)$ with

$$32^{80}+8^5\binom{10}{5}112^{75}*12^5$$

points. This yields $\mu(3)\geq 0.724851\ldots$.

Finally we discuss which asymptotic results are obtainable from Lemma 10 provided all needed $\tilde{I}_c(l,t)$ existed. With the above notation we have $|\tilde{I}_c(l,t)(A_0,\ldots,A_c)|=\binom{l}{t}N^{l-t}M^t$. Using the well known asymptotic relation $2^{lh(t/l)}\sim\binom{l}{t}$ between the binary entropy function $h(x):=-x\log_2(x)-(1-x)\log_2(1-x)$ and the binomial coefficients (see e.g. [11]), we see that we would asymptotically get

$$\mu(q)\geq\frac{1}{n}(h(t/l)\log_q(2)+((l-t)/l)\log_q(N)+t/l\log_q(M)).$$

The usual analytic procedure shows that at $l=t\frac{N+M}{M}$ we obtain the maximum and so would have:

$$\mu(q)\geq\frac{\log_q(N+M)}{n}.$$

For our ternary example it would therefore be possible to reach $\mu(3) \geq \frac{\log_3(124)}{6} = 0.731268\ldots$ if all $\tilde{I}_c((10\frac{1}{3})t, t)$ would exist.

This leaves us with the interesting research problem to construct $\tilde{I}_2(l,t)$, or at least large subsets of $\tilde{I}_2(l,t)$, in range of $l = (10\frac{1}{3})t$ for large t.

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